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Intersections of 1-forms and valuations in a local regular surface

Carlos Galindo[1]

Department of Mathematics, University Jaume I, C. Penyeta Roja, 12071 Castellon, Spain

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Abstract

We introduce intersection multiplicities between singular differential 1-forms in the plane and valuations centered at the plane. We prove a "Noether formula" showing the behaviour of the intersection multiplicity under quadratic transformations. Several methods to compute this number are derived. In the last part of this paper, we use intersection numbers to give a description of the desingularization of a 1-form without performing blow-ups.

1. Introduction

Let (R, m) be a regular 2-dimensional noetherian local ring with m the maximal ideal and algebraically closed residue field of characteristic zero. Assume that R contains a coefficient field K. Let us consider the scheme $X = \operatorname{Spec} R$ and denote P the closed point of X. Throughout the paper we will consider a differential 1-form $\eta \in \Omega_{R/K}$ and the foliation $\mathscr{F}$ given by it (as sub-$\mathcal{O}_X$-module of the sheaf of relative differentials $\Omega_{X/K}$ generated by η). It is well known [5, 9] that after finitely many successive quadratic transformations (blowing-ups in closed points) $\mathscr{F}$ becomes a foliation $\mathscr{F}'$ with (finitely many) singularities, all of them simple.

The relation between the valuations centered at R and the simple sequences of point blowing-ups of $\operatorname{Spec} R$ [10] will allow us to define an *intersection multiplicity* of differential forms and valuations. In this paper, we define (Definition 3.2) this value that generalizes the natural intersection multiplicity for a plane 1-differential form and an analytic (or formal) branch, we study the intersection multiplicity for each type of valuations (Section 2.2) and we prove a *Noether formula* (Theorem 3.3) for this number.

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Each valuation can be expressed by parametric equations which can be derived from its Hamburger–Noether expansion [7]. This fact is similar to the one for irreducible algebroid curves [4].

Now, let $P = P_0, P_1, \ldots, P_n$ be a simple sequence of infinitely near points, υ_i the valuation associated to $P_0, P_1, \ldots, P_i$, $0 \le i \le n$, and $\omega^{(i)}$ the germ in P_i of the strict transform $\mathscr{F}^{(i)}$ of $\mathscr{F}$ after blowing-up along $P_0, P_1, \ldots, P_{i-1}$. The computation of the intersection multiplicity between ω and υ_i is easy if υ_i is given by parametric equations. The sequence of these intersection numbers gives us a useful method to decide if $\omega^{(n)}$ is singular applying the Noether formula.

Intersection multiplicity sequences as above, also allow us to know whether P_n is a regular, a simple or a dicritical point, and moreover one can obtain the singular points of the foliation after blowing-up along P_n.

If f and g belong to a (commutative) ring with identity, the notation $f \nmid h$ will mean that f is not a factor of h in that ring. For unexplained notation we refer to [3].

2. Preliminaries

2.1. 1-forms over X

Let R be as in the Introduction and K a coefficient field for R. All the differentials we shall consider will be relative to the field K, so we will often omit the reference to K in the notations. We have two R-modules of differentials, the Kähler differentials Ω_R and the separated (Kähler) differentials $\bar{\Omega}_R$, i.e.,

$$\bar{\Omega}_R := \frac{\Omega_R}{\bigcap_{i=0}^{\infty} m^i \Omega_R},$$

and the corresponding derivations $\mathrm{d}: R \to \Omega_R$ (resp. $\bar{\mathrm{d}}: R \to \bar{\Omega}_R$). As for the couple (Ω_R, d), the couple $(\bar{\Omega}_R, \bar{\mathrm{d}})$ is characterized, up to an isomorphism, by the following universal property: if (N, δ) is a couple, where N is an m-adic separated R-module (i.e. $\bigcap_{i=0}^{\infty} m^i N = 0$) and $\delta: R \to N$ an R-derivation, then there exists a unique R-homomorphism $f: \bar{\Omega}_R \to N$ such that $f \circ \bar{\mathrm{d}} = \delta$. We also consider the $\hat{R}$-modules, $\Omega_{\hat{R}}$ and $\bar{\Omega}_{\hat{R}}$, $\hat{R}$ being the m-adic completion of R. In general, Ω_R, $\bar{\Omega}_R$ and $\Omega_{\hat{R}}$ are not free modules, however, $\bar{\Omega}_{\hat{R}}$ is always a rank two free $\hat{R}$-module. If $\{x, y\}$ is a regular system of parameters (rsp) of R, then $\{\bar{\mathrm{d}}x, \bar{\mathrm{d}}y\}$ is a basis for the $\hat{R}$-module $\bar{\Omega}_{\hat{R}}$.

If for some rsp $\{x, y\}$, $(\partial/\partial x)R \subset R$ and $(\partial/\partial y)R \subset R$, then $\bar{\Omega}_R$ is a finitely generated R-module and $\bar{\Omega}_R \otimes_R \hat{R} \cong \bar{\Omega}_{\hat{R}}$ as well (see [2]). This is the case, for instance, when R is a convergent power series ring or a localization of a finite type K-algebra. In both cases $\bar{\Omega}_R$ is also a rank two free module generated by $\{\bar{\mathrm{d}}x, \bar{\mathrm{d}}y\}$. Moreover, in the analytic case $\bar{\Omega}_R$ is isomorphic to the R-module of germs of holomorphic 1-forms, and in the algebraic case we have $\Omega_R = \bar{\Omega}_R$, since Ω_R is itself a rank two free R-module.

The free $\hat{R}$-module $\bar{\Omega}_{\hat{R}}$ can be viewed as the module of "formal" differential forms, i.e., it plays the same role as the completion $\hat{R}$ does for the functions. For this reason we shall use it to define formal germs for foliations. In the analytic (resp. algebraic) case, we can alternatively define analytic (resp. algebraic) germs without changes using the R-module $\bar{\Omega}_R$ (resp. Ω_R) instead of $\bar{\Omega}_{\hat{R}}$.

Consider a non-zero 1-form $\eta \in \Omega_R$ and let us consider its image $\bar{\eta}$ in $\bar{\Omega}_{\hat{R}}$ under the composition of the natural maps

$$\Omega_R \longrightarrow \bar{\Omega}_R \longrightarrow \bar{\Omega}_{\hat{R}}.$$

Let ω be the saturation of $\bar{\eta}$, i.e., if $\bar{\eta} = p\,\bar{\mathrm{d}}x + q\,\bar{\mathrm{d}}y$ and h the greatest common divisor of p and q, then $\omega = h^{-1}\bar{\eta} = a(x, y)\,\bar{\mathrm{d}}x + b(x, y)\,\bar{\mathrm{d}}y \in \bar{\Omega}_{\hat{R}}$. For the sake of simplicity of notations, when dealing with $\bar{\Omega}_{\hat{R}}$ we shall write $\mathrm{d}x$, $\mathrm{d}y$ instead of $\bar{\mathrm{d}}x$, $\bar{\mathrm{d}}y$. Note that ω is determined up to multiplication by a unit of $\hat{R}$ and it does not depend on the chosen rsp. The form ω will be referred as the formal germ (or simply germ if there is no confusion) at P of the foliation $\mathscr{F}$ given by η.

The multiplicity $v_P(\omega)$ (or $v_P(\mathscr{F})$) is defined to be the least of the ($\hat{m}$-adic) orders of the elements a and b, and it does not depend on the rsp. The point P is said to be regular for $\mathscr{F}$ if $v_P(\omega) = 0$ and P is said to be simple if $v_P(\omega) = 1$, some of the eigenvalues α, β of the linear map

$$\hat{m}/\hat{m}^2 \longrightarrow \hat{m}/\hat{m}^2$$

induced by the derivation $\delta = -b(\partial/\partial x) + a(\partial/\partial y)$, say β, is different from zero and $\alpha/\beta \notin \mathbb{Q}_+$ (=strictly positive rational numbers).

If $\gamma: \operatorname{Spec} K[[t]] \to X$ is an algebroid curve on X with origin at P, then we have the induced maps (denoted by γ^*, in general), $R \to K[[t]]$, $\hat{R} \to K[[t]]$ and $\bar{\Omega}_{\hat{R}} \to \bar{\Omega}_{K[[t]]}$, the later being the module of separated differentials for $K[[t]]$. The curve γ will be said to be an integral curve, or separatrix (see [3]), for $\mathscr{F}$ if $\gamma^*(\omega) = 0$, that is, if

$$a(x(t), y(t))x'(t) + b(x(t), y(t))y'(t) = 0,$$

where $x(t) = \gamma^*(x)$ and $y(t) = \gamma^*(y)$.

Now consider the blowing-up $\pi: X^{(1)} \to X$ of X at P and take a point P_1 in the exceptional divisor L_P. If $R_1 = \mathcal{O}_{X^{(1)}, P_1}$, we have the form $\eta_1 = \pi^*\eta$ (i.e., the image of η by the natural map $\Omega_R \to \Omega_{R_1}$) whose corresponding formal germ $\omega^{(1)} \in \bar{\Omega}_{\hat{R}_1}$ at P_1 will be called the germ of the strict transform $\mathscr{F}^{(1)}$ of $\mathscr{F}$ at P_1. The exceptional divisor L_P is said to be dicritical (resp. non-dicritical) if its germ at P_1 is not (resp. is) an integral curve for the strict transform of $\mathscr{F}$ for some (or, equivalently, for any) $P_1 \in L_P$. If a_0 (resp. b_0) is the homogeneous component of degree $v_P(\omega)$ in the Taylor expansion of a (resp. b), then L_P is dicritical if, and only if, $xa_0 + yb_0 = 0$.

Formal germs of strict transforms can be easily computed as strict transforms of formal germs. Indeed, assume that P_1 is the point corresponding to the tangent to the curve $y = \alpha x$, $\alpha \in K$. Then an rsp for R_1 is given by $x_1 = x$, $y_1 = (y - \alpha x)/x$, the induced morphism $\hat{R} = K[[x, y]] \to \hat{R}_1 = K[[x_1, y_1]]$ is given by $x \to x_1$,

$y \to x_1 y_1 + \alpha x_1$ and $\omega^{(1)}$ can be computed from ω by the expression

$$\omega^{(1)} = x^{-(v(\omega)+\varepsilon_P)}\{a(x_1, x_1 y_1 + \alpha y_1)\,dx_1$$

$$+ b(x_1, x_1 y_1 + \alpha y_1)(x_1\,dy_1 + y_1\,dx_1 + \alpha\,dx_1)\},$$

where $\varepsilon_P = 0$ if L_P is non-dicritical and $\varepsilon_P = 1$ otherwise.

Finally, we can make further blowing-ups, i.e., if we have a chain $P_0 = P, P_1, \ldots, P_n$, of infinitely near points to P, that is P_i is a point in the exceptional divisor L_i created by the blowing-up of the point P_{i-1}, then we have 2-dimensional local rings $R_i = \mathcal{O}_{X^{(i)}, P_i}$, where $X^{(0)} = X$ and $X^{(i)} = \mathrm{Bl}_{P_{i-1}} X^{(i-1)}$ for $i \geq 1$, and formal germs $\omega^{(i)} \in \bar{\Omega}_{\hat{R}_i}$ (of the strict transforms $\mathcal{F}^{(i)}$ of $\mathcal{F}$ at P_i) which can be computed from ω by formal substitutions.

The theorem of resolution of singularities for foliations (see [5] and [9]) asserts that after finitely many blow-ups at infinitely near points to P, we get a scheme X' such that the formal germs of the strict transform of $\mathcal{F}$ at all the closed points of X', are either regular or they have simple singularities. Note that, for instance, the proof in [5] can be adapted to the formal case without changes.

Also we have the following facts (see [6]):

(1) Only one (resp. two) smooth (resp. smooth and both meet transversely) integral curve passes through a regular (resp. simple) point.

(2) In the case of simple points appearing in the resolution, one (or may be two) of the integral curves are components of the exceptional divisor.

(3) By blowing-up regular or simple points, only regular and simple points can appear. Moreover, only non-dicritical divisors can appear. Thus, in particular, from the existence of resolution, only finitely many dicritical divisors can be created from P.

2.2. *Valuations centered in P*

Let (R, m) be as in the Introduction. Let F denote the field of fractions of R. There is a 1–1 correspondence between the set of *valuations of F centered in R* (valuations in the sequel) and the set of *simple sequences of quadratic transformations* (ssqt)

$$\pi: \cdots \longrightarrow X^{(n)} \xrightarrow[\pi_n]{} X^{(n-1)} \cdots \longrightarrow X^{(1)} \xrightarrow[\pi_1]{} X^{(0)} = X = \operatorname{Spec} R,$$

π_i centered in $P_{i-1} \in X^{(i-1)}$, $P_0 = P$, [10]. Let υ be a valuation and R_υ its valuation ring. Using the Abhyankar inequalities [1]:

$$\mathrm{rk}\,\upsilon + \mathrm{tr.deg}\,\upsilon \leq \dim R \quad \text{and} \quad \mathrm{rat.rk}\,\upsilon + \mathrm{tr.deg}\,\upsilon \leq \dim R,$$

we can make a table that classifies the valuations by *rank* (rk), *rational rank* (rat.rk) and *transcendence degree* (tr.deg) of the valuation υ; see Table 1.

Table 1

Type	Rank	Rational rank	Transcendence degree	Discrete	Value group Γ
A	1	1	1	yes	$\Gamma = \mathbb{Z}$
B	2	2	0	yes	$\Gamma = \mathbb{Z} \oplus \mathbb{Z}$
C	1	1	0	yes	$\Gamma = \mathbb{Z}$
D	1	2	0	not	$\Gamma \subset \mathbb{R}$, $\Gamma \not\subset \mathbb{Q}$
E	1	1	0	not	$\Gamma \subset \mathbb{Q}$, $\Gamma \not\subset \mathbb{Z}$

The valuations of type A are said to be *divisorial valuations* and are associated to finite ssqt, [10].

Let $\{x, y\}$ be the above rsp of m. Associated to each valuation v is an expression called *Hamburger–Noether expansion* (HNE), that provides complete information about the infinitely near points P_i, as follows:

$$\begin{aligned}
&y = a_{01}x + a_{02}x^2 + \cdots + a_{0h_0}x^{h_0} + x^{h_0}z_1,\\
&x = z_1^{h_1}z_2,\\
&\vdots\\
&z_{s_1-2} = z_{s_1-1}^{h_{s_1-1}}z_{s_1},\\
D:\quad&\\
&z_{s_1-1} = a_{s_1k_1}z_{s_1}^{k_1} + \cdots + a_{s_1h_{s_1}}z_{s_1}^{h_{s_1}} + z_{s_1}^{h_{s_1}}z_{s_1+1},\\
&\vdots\\
&z_{s_p-1} = a_{s_pk_p}z_{s_p}^{k_p} + \cdots + a_{s_ph_{s_p}}z_{s_p}^{h_{s_p}} + z_{s_p}^{h_{s_p}}z_{s_p+1},\\
&\vdots\\
&h_m \in \mathbb{N};\ s_q \in \mathbb{N};\ k_q \in \mathbb{N},\ 2 \le k_q \le h_{s_q};\ a_{s_qj} \in K;\ a_{s_qk_q} \neq 0.
\end{aligned}$$

This expansion is infinite except for the case A, whose last row will be as:

$$z_{s_g-1} = a_{s_gk_g}z_{s_g}^{k_g} + \cdots + a_{s_gh_{s_g}}z_{s_g}^{h_{s_g}} + z_{s_g}^{h_{s_g}+1}U, \quad U \in F\backslash R,\ v(U) = 0,$$

and U is written to complete the equality (see [7] for details).

The change of row is related to the change of position of the blowing-up center. There are three kinds of positions:

(a) The center is not an intersection point of exceptional curves.

(b) The center is an intersection point of the two last created exceptional curves.

(c) The center is an intersection point of the last exceptional curve and another curve different from the next to the last one.

The center is called a *free point* in the case (a) and a *satellite point* otherwise. This expansion is similar to the irreducible plane algebroid curves [4].

2.3. Parametrization of valuations

We are going to obtain parametric equations of the valuation υ from its HNE. We obtain the equations by considering the results of Spivakovsky [10].

Valuation of type A. Let us put $z_{s_g} = t$. We obtain the equations $x = x(t, U)$, $y = y(t, U)$ in $K[[t, U]]$, and if $h \in R$, then,

$$\upsilon(h) = \mathrm{ord}_t\, h(x(t, U), y(t, U)),$$

where ord_t denotes the normalized natural valuation of $K((U))((t))$.

Valuation of type B *and* C. The last row is $z_{s_g - 1} = a_{s_g k_g} z_{s_g}^{k_g} + \cdots$ (infinite sum). If t denotes z_{s_g}, the parametric equations of υ in $K[[t]]$ will be $x = x(t)$, $y = y(t)$. The parametrization morphism $p: R \to K[[t]]$ defined $p(x) = x(t)$; $p(y) = y(t)$ can be extended to a morphism $\hat{p}$: $\hat{R} \to K[[t]]$ of the m-adic completion, $\hat{R}$, of R, with the same definition. If $\mathrm{Ker}\, p = (g)$, the valuation υ is of type C if, and only if, $g = 0$. Moreover, if $\mathrm{Ker}\, \hat{p} = (f)$, one has for $h \in R$,

$$\upsilon(h) = \begin{cases} (\alpha, \mathrm{ord}_t\, \bar{h}(x(t), y(t)) & \text{if } h = f^{\alpha}\bar{h};\ f \nmid \bar{h}, \\ \mathrm{ord}_t\, h(x(t),\ y(t)) & \text{if } f \nmid h, \end{cases}$$

where ord_t denotes the normalized natural valuation of $K((t))$.

Valuation of type D. Now, there is a row, s_g, in its HNE such that the rows appearing after s_g are related to satellite points. We shall write $K\langle t\rangle$ for the ring of formal series $\sum_{r \in \mathbb{R}} a_r t^r$, $a_r \in K$, such that the set $\{r \in \mathbb{R} \mid a_r \neq 0\}$ is well-ordered (i.e., every subset contains its minimal element). If we denote, γ, the infinite continued fraction, $\langle h_{s_g}, h_{s_g + 1}, \ldots\rangle \in \mathbb{R}\backslash\mathbb{Q}$ and $z_{s_g + 1} = t$ and we choose $z_{s_g} = t^{\gamma}$, then the HNE gives parametric equations $x = x(t)$, $y = y(t)$ in $K\langle t\rangle$. For $h \in R$, we have $\upsilon(h) = \mathrm{ord}_t\, h(x(t), y(t))$.

Valuation of type E. The structure of D in Section 2.2 is repeated indefinitely. Let us consider, in the HNE, for each index i, all the rows until the s_i row and let us omit $z_{s_i}^{h_{s_i}} z_{s_i + 1}$, then we obtain the HNE for an irreducible algebroid curve. The relation between the HNE and the Puiseux expansion for this curve [4] gives us equations $y = \sum a_{ir} x^r$, where a_{ir} are constants in K independent of i that can be chosen inductively. This procedure gives parametric equations $x = x(t) = t$, $y = y(t) \in K\langle t\rangle$; $y(t) = \sum a_r t^r$, $r \in \mathbb{Q}$. There are infinite values r with $a_r \neq 0$ and the sequence of denominators of them, as quotient of relatively prime elements, is not bounded. For $h \in R$, we have $\upsilon(h) = \mathrm{ord}_t\, h(x(t), y(t))$.

We have written ord_t for the normalized natural valuation of the fraction field of $K\langle t\rangle$ for valuations of types D and E.

Usually, we shall write the valuations through parametric equations over a field extension K' of K as $x = x(t)$, $y = y(t)$ in $K'\langle t\rangle$, where $x(t)$, $y(t)$ are not units of $K'\langle t\rangle$ and at least one of them is different from zero. K' will be the algebraic closure of $K(U)$

for divisorial valuations and $K' = K$ otherwise. Here, and in the sequel, ord_t is the normalized natural valuation of the fraction field of $K'\langle t\rangle$. If we consider the parametrization morphism, $\hat{p}: K[[x, y]] \to K'\langle t\rangle$, defined by $\hat{p}(x) = x(t)$, $\hat{p}(y) = y(t)$; the unique valuation, $\hat{\upsilon}$, which extends υ to the field of fractions of $\hat{R}$ will be

$$\hat{\upsilon}(h) = \begin{cases} (\alpha, \mathrm{ord}_t \bar{h}(x(t), y(t)) & \text{if } \mathrm{Ker}\,\hat{p} = (f) \text{ and } h = f^\alpha \bar{h},\ f \nmid \bar{h}, \\ \mathrm{ord}_t h(x(t), y(t)) & \text{if } \mathrm{Ker}\,\hat{p} = 0, \end{cases}$$

for $h \in K[[x, y]]$.

Remark. We can obtain the HNE of υ from the parametric equations of a valuation υ through the Hamburger–Noether algorithm. Indeed, suppose $\mathrm{ord}_t y \geq \mathrm{ord}_t x$ where $\geq$ denotes the natural order in $\mathbb{R}$, or the lexicographical order in $\mathbb{Z}^2$. Now, let us divide $y/x = a_{01} + y_1$. If $a_{01} \notin K$, the algorithm ends, and we put $y_1 = U$. Otherwise if $\mathrm{ord}_t y_1 \geq \mathrm{ord}_t x$, we divide y_1/x and, if $\mathrm{ord}_t y_1 < \mathrm{ord}_t x$, we let $z_1 = y_1$ and divide x/y_1, and so on.

3. Intersection multiplicity

Let ω and υ as above. For valuations of types B, C, D and E the parametrization morphism can be written as $p: R \to K\langle t\rangle$ and it induces an R-modules morphism between the modules of separated differentials, $p_*: \bar{\Omega}_{R/K} \to \bar{\Omega}_{K\langle t\rangle/K}$. More precisely, for any given $z \in K\langle t\rangle$, $z = \sum_{r \in I \subseteq \mathbb{R}} a_r t^r$, we put $z' = \sum_{r \in I \subseteq \mathbb{R}} r a_r t^{r-1}$. Hence, we have

$$p_*(\omega) = [a(x(t), y(t))x'(t) + b(x(t), y(t))y'(t)]\,\mathrm{d}t.$$

When the valuation υ is of type A, the parametrization morphism will be $p: R \to K[[t, U]]$ with $p(x) = x(t, U)$ and $p(y) = y(t, U)$. Then $p_*: \bar{\Omega}_{R/K} \to \bar{\Omega}_{K[[t,U]]/K}$ is given by

$$\begin{aligned} p_*(\omega) = &\left[a(x(t, U), y(t, U))\frac{\partial x(t, U)}{\partial t} + b(x(t, U), y(t, U))\frac{\partial y(t, U)}{\partial t}\right]\mathrm{d}t \\ &+ \left[a(x(t, U), y(t, U))\frac{\partial x(t, U)}{\partial U} + b(x(t, U), y(t, U))\frac{\partial y(t, U)}{\partial U}\right]\mathrm{d}U. \end{aligned}$$

Definition 3.1. Let γ be an irreducible algebroid plane curve in P defined by $h \in \hat{R}$. Consider $\mathcal{O} = \hat{R}/(h)$ its local ring and $\bar{\mathcal{O}} = K[[t]]$ its integral closure. Put $\sigma: \mathcal{O} \to K[[t]]$ the morphism that a primitive parametrization of γ gives us. This one induces $\sigma_*: \bar{\Omega}_{\mathcal{O}/K} \to \bar{\Omega}_{K[[t]]/K}$. The intersection multiplicity between the germ ω and the curve γ is $(\omega, \gamma)_P = \mathrm{ord}_t[\sigma_*(\bar{\omega})]$ ($\bar{\omega}$ is the element of $\bar{\Omega}_{\mathcal{O}/K}$ that ω yields). If $\sigma_*(\bar{\omega}) = 0$ we put $(\omega, \gamma)_P = \infty$. From the definition, $(\omega, \gamma)_P = 0$ if, and only if, ω is regular and γ is smooth and transversal to ω. The set of multiplicities between ω and the family of curves as above generates a semigroup [8]. In the sequel, the curve defined by $h \in \hat{R}$ will also be denoted by h.

Definition 3.2. The intersection multiplicity between υ and ω is defined to be the number

$$(R_\upsilon, \omega) = \begin{cases} \left(0, \dfrac{\operatorname{ord}_t p_*(\omega)}{\min\{\operatorname{ord}_t p(x), \operatorname{ord}_t p(y)\}}\right) \in \mathbb{Z} \oplus \mathbb{R}, & \text{if } p_*(\omega) \neq 0, \\ (1, 0) \in \mathbb{Z} \oplus \mathbb{R}, & \text{if } p_*(\omega) = 0. \end{cases}$$

Let υ be a valuation, π the sequence of quadratic transformations of Section 2.2 associated to it and put $\omega = \omega^{(0)}$. As in Section 2.1, m_i will denote the maximal ideal of the local ring $\mathcal{O}_{X^{(i)}, P_i}$, $\omega^{(i)}$ the germ of $\mathcal{F}^{(i)}$, the transformed foliation of $\mathcal{F}$ under $\pi_1 \pi_2 \cdots \pi_i$, in P_i, $\upsilon_{P_i}(\omega^{(i)})$ its algebraic multiplicity and L_i the exceptional divisor of π_i in $X^{(i)}$. L_{i+1} will be called *dicritical (exceptional) divisor* if P_i is a dicritical singularity and *non-dicritical* if P_i is not a dicritical singularity. Moreover, we write $\varepsilon^{(i)} = 0$ if P_i is a non-dicritical point and, $\varepsilon^{(i)} = 1$ otherwise. Then we have the following theorem.

Theorem 3.3 (Noether formula). *The following formula holds:*

$$(R_\upsilon, \omega) = \frac{1}{\upsilon(m)} \sum_{i=0}^{p} [\upsilon_{P_i}(\omega^{(i)}) + \varepsilon^{(i)}]\, \upsilon(m_i),$$

where $p = n$ if υ is divisorial and P_n the last blowing-up center and $p = \infty$ for the other types of valuations. If the series diverges, we will write $(R_\upsilon, \omega) = (1, 0)$.

Proof. Since $\upsilon(m) = \min\{\operatorname{ord}_t p(x), \operatorname{ord}_t p(y)\}$, we have to prove that, $\operatorname{ord}_t p_*(\omega) = \sum_{i=0}^{\infty (n)} [\upsilon_{P_i}(\omega^{(i)}) + \varepsilon^{(i)}]\, \upsilon(m_i)$. For all $i \in \mathbb{N}$ (respectively, $i \leq n$ whenever υ is divisorial) the valuation υ is, also, associated to the ssqt,

$$\cdots \to X^{(n)} \to X^{(n-1)} \to \cdots \to X^{(i+1)} \to \operatorname{Spec} \mathcal{O}_{X^{(i)}, P_i}.$$

Let D be the HNE of υ as in Section 2.2. Let $D^{(1)}$ be the HNE of υ for the rsp $\{x, y' = (y - a_{01}x)/x\}$ of $\mathcal{O}_{X^{(1)}, P_1}$. $D^{(1)}$ is given by

$$y' = a_{02}x + \cdots + a_{0h_0}x^{h_0 - 1} + x^{h_0 - 1}z_1,$$
$$\vdots$$
$$z_{s_p - 1} = a_{s_p k_p} z_{s_p}^{k_p} + \cdots + a_{s_p h_{s_p}} z_{s_p}^{h_{s_p}} + z_{s_p}^{h_{s_p}} z_{s_p + 1},$$
$$\vdots$$

and one has a parametrization morphism $p^{(1)}: \mathcal{O}_{X^{(1)}, P_1} \to K\langle t \rangle$ (resp. $K[[t, U]]$ if υ is of type A). The blowing-up formula of the germ ω [3, II.1] shows that,

$$\operatorname{ord}_t p_*(\omega) = \operatorname{ord}_t [x(t)^{\upsilon_{P_0}(\omega^{(0)}) + \varepsilon^{(0)}} p_*^{(1)}(\omega^{(1)})]$$

and we conclude $\mathrm{ord}_t p_*(\omega) = [v_{P_0}(\omega^{(0)}) + \varepsilon^{(0)}]\, v(m_0) + \mathrm{ord}_t p_*^{(1)}(\omega^{(1)})$, because $v(m_0) = \mathrm{ord}_t x(t)$ (respectively $x(t, U)$, if v is divisorial).

To complete the proof, we consider each type of valuation separately.

Valuations A. By induction, we are led to the initial stage $n = 0$. The HNE will be $y = xU$ and the parametric equations $x = t$; $y = Ut$, then,

$$p_*(\omega) = t^{v_P(\omega)}([a_{v_P(\omega)}(1, U) + Ub_{v_P(\omega)}(1, U) + \cdots]\,\mathrm{d}t + [tb_{v_P(\omega)}(1, U) + \cdots]\,\mathrm{d}U)$$

and $\mathrm{ord}_t p_*(\omega) = v_P(\omega) + \varepsilon^{(0)}$.

In the remaining types of valuations, one has for all $i = 0, 1, \ldots$

$$\mathrm{ord}_t p_*^{(i)}(\omega^{(i)}) = [v_{P_i}(\omega^{(i)}) + \varepsilon^{(i)}]\, v(m_i) + \mathrm{ord}_t p_*^{(i+1)}(\omega^{(i+1)}). \tag{3.3}$$

Now, we prove that there exist two possibilities:

(i) $\mathrm{ord}_t p_*^{(i)}(\omega^{(i)}) = (1, 0)$ for all i. Then the series $\sum_{i=0}^{\infty} [v_{P_i}(\omega^{(i)}) + \varepsilon^{(i)}]\, v(m_i)$ diverges and we write $(1, 0)$ for its sum.

(ii) $\lim_{i\to\infty} \mathrm{ord}_t p_*^{(i)}(\omega^{(i)}) = 0$, then $\sum_{i=0}^{\infty} [v_{P_i}(\omega^{(i)}) + \varepsilon^{(i)}]\, v(m_i)$ converges and its sum is $v(m)\,(R_v, \omega)$ by Eq. (3.3).

Valuations B *or* C. In a good rsp the irreducible algebroid curve f of Section 2.3 and v have the same parametric equations. If f is a separatrix of ω, and $f^{(i)}$ is the irreducible algebroid curve associated to v in the ring $\mathcal{O}_{X^{(i)}, P_i}$, then $f^{(i)}$ is a separatrix of $\omega^{(i)}$, hence $\mathrm{ord}_t p_*^{(i)}(\omega^{(i)}) = (1, 0)$ for all i. Moreover, for i large enough $\omega^{(i)}$ is desingularized (i.e. it remains only simple singularities). The separatrix $f^{(i)}$ is smooth, transverse to the divisor and it does not pass through a corner (see [3]). P_i will be the intersection between a separatrix and the divisor, which is a separatrix, since only a finite number of dicritical divisors appears in the desingularization and subsequent blowing-ups, [3]. Therefore P_i is simple. Thus there is i_0 such that for $i \geq i_0$, $\omega^{(i)}$ is singular and P_i is a simple point, therefore $(v_{P_i}(\omega^{(i)}) + \varepsilon^{(i)})\, v(m_i) = 1$ and the series $\sum_{i=0}^{\infty} [v_{P_i}(\omega^{(i)}) + \varepsilon^{(i)}]\, v(m_i)$ diverges. If f is not separatrix of ω, for i big enough, P_i is a regular point of $\omega^{(i)}$ and $f^{(i)}$ gives a smooth irreducible algebroid plane curve transversal to $\omega^{(i)}$ in P_i. Then $(R_v, \omega^{(i)}) = 0$ and therefore $\mathrm{ord}_t p_*^{(i)}(\omega^{(i)}) = 0$.

Valuations D. Let P_{i_0} be an infinitely near point of v such that the transformed of ω in P_{i_0} is not singular and such that all the infinitely near points of v after P_i are simple and satellite. This is so, since for $i \geq i_0$ all the points P_i are the intersection of two transversal non-dicritical divisors.

Let $v^{(i_0)}$ be the normalization of v such that $v^{(i_0)}(m_{i_0}) = 1$. Let us consider $\{x, y\}$ an rsp of m_{i_0} with $v^{(i_0)}(x) = 1$ and $v^{(i_0)}(y) = \alpha \in \mathbb{R}\setminus\mathbb{Q}$, then the equations of $v^{(i_0)}$ are $x = t$; $y = t^{\alpha}$. Let us write $\alpha = \langle c_1, c_2, \ldots\rangle$ as a continued fraction, $c_j \in \mathbb{N}\setminus\{0\}$, then $\alpha_{j-1} = c_j\alpha_j + \alpha_{j+1}$; $0 < \alpha_{j+1} < \alpha_j$; $\alpha_j \in \mathbb{R}$; $j = 1, 2, \ldots$; $\alpha_0 = \alpha$; $\alpha_1 = 1$. The equations for the normalization of v, $v^{(i)}$, such that $v^{(i)}(m_i) = 1$ are $x_i = t$, $y_i = t^{\alpha(i)}$ for $i \geq i_0$; moreover there exists $j = j(i)$ that increases with i such that $\alpha_j/\alpha_{j+1} \geq \alpha(i) \geq 1$. Furthermore, the germ $\omega^{(i)}$ can be written as $\omega^{(i)} = a(x_i, y_i)y_i\,\mathrm{d}x_i + b(x_i, y_i)x_i\,\mathrm{d}y_i$,

$a(0,0) \neq 0$ or $b(0,0) \neq 0$, because P_i is a simple satellite point. Therefore,

$$\operatorname{ord}_t p_*^{(i)}(\omega^{(i)}) = (R_{v^{(i)}}, \omega^{(i)}) = \alpha(i).$$

Now to prove (ii) it suffices to check that $\lim_{j\to\infty} \alpha_j = 0$. If we suppose $\lim_{j\to\infty} \alpha_j \neq 0$, then there exists $\varepsilon > 0$ and a fixed index j_0 such that $\alpha_j \geq \varepsilon$ for all indices $j \geq j_0$. Consider $l \in \mathbb{N}$ such that $l\varepsilon > \alpha$, then if we put $q = l + j_0$ we have $\alpha_{q-1} \geq c_q\alpha_q + \alpha_{q+1} \geq 2\varepsilon$; also, $\alpha_{q-2} \geq c_{q-1}\alpha_{q-1} + \alpha_q \geq 3\varepsilon; \dots$ In conclusion, $\alpha_{j_0+1} \geq l\varepsilon > \alpha$, which is a contradiction.

Valuations E. There is $i_0 \in \mathbb{N}$ such that P_{i_0} is an infinitely near point of υ which is a non-simple and non-singular point of $\omega^{(i_0)}$, and such that for $i > i_0$ if P_i is a satellite point it is also a simple point. Then, for $i \geq i_0$, $(v_{P_i}(\omega^{(i)}) + \varepsilon^{(i)}) = 0$ if P_i is not a satellite point and, $(v_{P_i}(\omega^{(i)}) + \varepsilon^{(i)}) = 1$ if P_i is a satellite point. Let $\underline{\upsilon}$ be the normalization of υ such that $\underline{\upsilon}(m) = 1$ and $\{x, y\}$ an rsp of m, so that $\underline{\upsilon}(x) = 1$ and $\underline{\upsilon}(y) > 1$. The HNE shows the following expressions (see Section 2.3.E):

$$\begin{aligned} &\alpha_{s_{l-1}} = \alpha_{s_{l-1}+1} h_{s_{l-1}+1} + \alpha_{s_{l-1}+2}, && 0 < \alpha_{s_{l-1}+2} < \alpha_{s_{l-1}+1}, \\ &\alpha_{s_{l-1}+1} = \alpha_{s_{l-1}+2} h_{s_{l-1}+2} + \alpha_{s_{l-1}+3}, && 0 < \alpha_{s_{l-1}+3} < \alpha_{s_{l-1}+2}, \\ &\alpha_{s_{l-1}} = \alpha_{s_l} k_l, \end{aligned} \tag{3.3.E}$$

for $1 \leq l \leq g$, g as big as we want, and $\alpha_{s_0} = \alpha_0 = 1$. As above, the computation of $\operatorname{ord}_t p_*^{(i)}(\omega^{(i)})$ can be restricted to satellite points because the (3.3) formula proves that this value does not change when i corresponds to a free point with respect to the previous satellite point. Now, P_i being simple and satellite point implies that $\operatorname{ord}_t p_*^{(i)}(\omega^{(i)}) = \alpha_{i-1}$ for $\underline{\upsilon}(z_i) = \alpha_i$ and $\{z_{i-1}, z_i\}$ the rsp of m_i that the HNE gives us. Using (3.3.E) there exists j, that increases with i, such that $\alpha_{s_{j+1}} \leq \alpha_{i-1} \leq \alpha_{s_j}$. Therefore, it is sufficient to prove that $\lim_{j\to\infty} \alpha_{s_j} = 0$. If not, then there exists $\varepsilon > 0$ and a fixed index j_0 with $\alpha_{s_j} \geq \varepsilon$ for all $j \geq j_0$. Let $m \in \mathbb{N}$ be such that $2^m\varepsilon > 1$ and $m \geq j_0$, then $1 = \alpha_{s_0} \geq 2\alpha_{s_1} \geq 2^2\alpha_{s_2} \geq \cdots \geq 2^m\alpha_{s_m} > 1$ which is a contradiction. This completes the proof. □

Remark. As a result of the above theorem, $p_*(\omega) = 0$ is possible only for valuations of type B or C. When $p_*(\omega) \neq 0$, we shall consider (R_υ, ω) as element of $\mathbb{R}$, since its first component is zero. Moreover the intersection multiplicity is independent of the normalization of υ. It is also worth mentioning that $(R_\upsilon, \omega) \in \mathbb{Q}$, for valuations of type E (see [7] for details).

Intersection multiplicity with divisorial valuations

Let υ be a divisorial valuation and P_n the last blowing-up center of the ssqt associated to υ. The Noether formula proves that $(R_\upsilon, \omega) \in \mathbb{Q}$. If $\upsilon^{(n)}$ is the normalization of υ such that $\upsilon^{(n)}(m_n) = 1$, we have $\upsilon^{(n)}(m)(R_\upsilon, \omega) \in \mathbb{N}$ and we call it the intersection number between υ and ω, $N(R_\upsilon, \omega)$.

Let ζ be the generic point of the center of υ in the scheme $X^{(n+1)}$ and $\mathscr{U}$ any of the $n+2$ affine charts that cover $X^{(n+1)}$. If $P_{n+1} \in L_{n+1} \bigcap \mathscr{U}$ is any closed point and $\eta_{L_{n+1}}$ is the generic point of L_{n+1} in $\operatorname{Spec}(\mathcal{O}_{X^{(n+1)}, P_{n+1}})$ then $(\mathcal{O}_{X^{(n+1)}, P_{n+1}})_{\eta_{L_{n+1}}} \cong R_\upsilon$. We may consider the natural morphism $j: \operatorname{Spec} R_\upsilon \to X^{(n+1)}$ and its induced morphism $\Omega_{X,P/K} \to \Omega_{R_\upsilon/K}$. Again the R_υ-module $\Omega_{R_\upsilon/K}$ is not free, so one can pass to separated differentials; i.e. one has an induced morphism:

$$j^*\pi^*: \bar{\Omega}_{\hat{R}/K} \to \bar{\Omega}_{\hat{R}_\upsilon/K} = \left(\frac{\Omega_{\hat{R}_\upsilon/K}}{\bigcap_{i=0}^{\infty} m_\upsilon^i \bar{\Omega}_{\hat{R}_\upsilon/K}}\right),$$

m_υ being the maximal ideal for the valuation ring $\hat{R}_\upsilon$.

Now, $\bar{\Omega}_{\hat{R}_\upsilon/K}$ is a rank two free module. In fact, the residual field of R_υ (and also that of $\hat{R}_\upsilon$) is isomorphic to a 1-variable function field $K(U)$, so if we take a uniformizing t, we have a K-algebra isomorphism

$$\hat{R}_\upsilon \cong K(U)[[t]].$$

From this equivalence, we conclude that we have two derivations $\partial/\partial U$, $\partial/\partial t$ taking $\hat{R}_\upsilon$ into itself. Now set $M = \hat{R}_\upsilon \oplus \hat{R}_\upsilon$ and $\mathscr{D}: \hat{R}_\upsilon \to M$ the mapping given by $\mathscr{D}(g) = (\partial g/\partial U,\ \partial g/\partial t)$. The couple $(M, \mathscr{D})$ satisfies that M is an m_υ-separated $\hat{R}_\upsilon$-module and $\mathscr{D}$ a derivation. We claim that this couple has the universal property for the separated Kähler differentials. To prove the claim, let us take a couple (N, δ) where N is an m_υ-separated $\hat{R}_\upsilon$-module and $\delta: \hat{R}_\upsilon \to N$ a derivation. Define an $\hat{R}_\upsilon$-morphism $f: M \to N$ by $f(1,0) = \delta U$, $f(0,1) = \delta t$. We shall check that $f \circ \mathscr{D} = \delta$. Indeed, if $g \in \hat{R}_\upsilon$ we must check the equality

$$\delta g - \frac{\partial g}{\partial U}\,\delta U - \frac{\partial g}{\partial t}\,\delta t = 0.$$

In order to do it, take an integer $i \geq 0$, and let g_i be an element of $K(U)[t]$ such that $\operatorname{ord}_t(g - g_i) \geq i + 1$. Since g_i is a rational function in t and U, we have, by Leibniz rule for derivatives

$$\delta g_i - \frac{\partial g_i}{\partial U}\,\delta U - \frac{\partial g_i}{\partial t}\,\delta t = 0,$$

hence

$$\delta g - \frac{\partial g}{\partial U}\,\delta U - \frac{\partial g}{\partial t}\,\delta t \in t^i N = m_\upsilon^i N.$$

Since $\bigcap_{i=0}^{\infty} m_\upsilon^i N = 0$, the left member of the above equality is 0.

Thus the claim is shown. Now, it follows that $\bar{\Omega}_{\hat{R}_\upsilon/K} \cong M$. Therefore $\bar{\Omega}_{\hat{R}_\upsilon/K}$ is a rank two free module as required.

Let $\bar{\omega} \in \bar{\Omega}_{\hat{R}_\upsilon/K}$. If (a, b) are its coordinates in a basis, $a, b \in \hat{R}_\upsilon$, we define the value $\underline{\upsilon}(\bar{\omega})$ to be $\underline{\upsilon}(\bar{\omega}) = \min\{\operatorname{ord}_t(a), \operatorname{ord}_t(b)\}$. This value is independent of the basis. Then we define the υ-normalized valuation of the germ ω to be $\underline{\upsilon}(\omega) = \underline{\upsilon}(j^*\pi^*(\omega))$.

Definition 3.4. A element, h, of the m-primary ideal associated to υ is said to be *general* if h is an analytically irreducible plane curve and its strict transform under the sequence π in $X^{(n+1)}$ is regular and transversal to L_{n+1}.

Theorem 3.5. *Let υ be a divisorial valuation. Then, the following natural numbers coincide*:

(i) $\min\{(\omega, h)_P \mid h$ *is the branch that gives a general element of* $\upsilon\} = (\omega, \upsilon)_P$.

(ii) $N(R_\upsilon, \omega)$.

(iii) $\underline{\upsilon}(\omega)$.

(iv) (*If* $x = x(t, U)$; $y = y(t, U)$ *are the equations from Section* 2.3 *for* υ)

$$\mathrm{ord}_t\left[a(x(t,U), y(t,U))\frac{\partial x(t,U)}{\partial t} + b(x(t,U), y(t,U))\frac{\partial y(t,U)}{\partial t}\right] := (D, \omega),$$

where U is the formal expression $\alpha + \beta t$.

Proof. (1) The HNE, $D(D^{(1)})$ of υ associated to R (to $\mathcal{O}_{X^{(1)}, P_1}$, respectively), in a suitable rsp, gives a bijection between the general elements h of υ associated to R and their strict transforms $h^{(1)}$, which are also general elements of υ associated to $\mathcal{O}_{X^{(1)}, P_1}$. The formula of blowing-up [3] and the equality $\upsilon^{(n)}(m) = e(h)$, where $e(h)$ denotes the algebraic multiplicity of the branch h [4, 3.3.2], prove that

$$(\omega, \upsilon)_P = (v_{P_0}(\omega^{(0)}) + \varepsilon^{(0)})\upsilon(m) + (\omega^{(1)}, \upsilon)_{P_1}.$$

(2) $N(R_\upsilon, \omega) = (v_{P_0}(\omega^{(0)}) + \varepsilon^{(0)})\upsilon(m) + N(R_\upsilon, \omega^{(1)})$, by the Noether formula.

(3) Let us consider an rsp of R such that $x = 0$ has not the direction defined by P_1, then $\pi_1^*(\omega) = x^{v_{P_0}(\omega^{(0)}) + \varepsilon^{(0)}}\omega^{(1)}$. If $\pi^{(2)} = \pi_2\pi_3 \cdots \pi_n$ then $j^*\pi^*(\omega) = x^{v_{P_0}(\omega^{(0)}) + \varepsilon^{(0)}} j^*\pi^{(2)*}(\omega^{(1)})$, thus,

$$\underline{\upsilon}(\omega) = [v_{P_0}(\omega^{(0)}) + \varepsilon^{(0)}]\,\upsilon^{(n)}(m) + \underline{\upsilon}(\omega^{(1)}).$$

(4) An argument similar to (1) proves that

$$(D, \omega) = (v_{P_0}(\omega^{(0)}) + \varepsilon^{(0)})\mathrm{ord}_t\, x(t, U)|_{U=\alpha+\beta t}) + (D^{(1)}, \omega^{(1)}).$$

The above arguments end the proof by induction once we prove the theorem for $n = 0$. The general elements of υ are the smooth plane irreducible algebroid curves in P_0 and then $(\omega, \upsilon)_P = v_{P_0}(\omega^{(0)}) + \varepsilon^{(0)}$. The Noether formula proves $N(R_\upsilon, \omega) = v_{P_0}(\omega^{(0)}) + \varepsilon^{(0)}$. On the other hand,

$$\underline{\upsilon}(\omega) = \underline{\upsilon}(j^*\pi_1^*\omega) = \underline{\upsilon}(j^* x^{v_{P_0}(\omega^{(0)}) + \varepsilon^{(0)}}\omega^{(1)}) = v_{P_0}(\omega^{(0)}) + \varepsilon^{(0)},$$

because $\underline{\upsilon}(j^*\omega^{(1)}) = 0$ since we can take $P_1 = P_{n+1}$ the point of L_1 that corresponds to the direction of the x-axis. If we calculate (D, ω), we obtain the multiplicity of the germ ω with the smooth irreducible algebroid curve in P with generic tangent whose strict transformed has, also, generic tangent and then $(D, \omega) = v_{P_0}(\omega^{(0)}) + \varepsilon^{(0)}$. $\square$

Remark. Statement (iv) of the above theorem is false if we take $U \in K$ because some dicritical center might appear in the desingularization of ω. For instance, if $\omega = 2y\,dx - x\,dy$, the HNE is $y = x^2 U$ and $(D, \omega) = \operatorname{ord}_t(2Ut^2 - 2Ut^2)$ which is not defined.

Statement (iv) is of interest because it gives us a powerful method to compute the number $N(R_\upsilon, \omega)$. In the sequel, we write $\omega(D)$ for the expression

$$a(x(t, U), y(t, U))\frac{\partial x(t, U)}{\partial t} + b(x(t, U), y(t, U))\frac{\partial y(t, U)}{\partial t}\bigg|_{U = \alpha + \beta t}.$$

Now we extend the above results to valuations B and C. The value $\underline{\upsilon}(\omega)$ cannot be defined because $\bar{\Omega}_{\hat{R}_\upsilon}$ is not an $\hat{R}_\upsilon$-module of finite type.

Intersection multiplicity with valuations of type B *and* C

Let υ be a B or C valuation with value group $\Gamma \subset \mathbb{Z}^2$. The product in $\mathbb{Q}$ induces a product in the set $\{0\} \oplus \mathbb{Q}$ that gives a normalization $\underline{\upsilon}$ of υ such that $\underline{\upsilon}(m) = (0, 1)$. If $*: \mathbb{Z} \times \mathbb{Z} \oplus \mathbb{Q} \to \mathbb{Z} \oplus \mathbb{Q}$ is the operation $m*(m', a) = (m', ma)$, we may define the intersection number between the germ ω and the valuation υ to be $N(R_\upsilon, \omega) = n*(R_\upsilon, \omega)$, where $n = e(f)$ is the algebraic multiplicity of the irreducible algebroid curve f of Section 2.3. Let us write $\upsilon^* = n*\underline{\upsilon}$. According to Theorem 3.3 one has the following theorem.

Theorem 3.6. *Let $\upsilon = \upsilon^*$ be a B or C valuation normalized as above. The following numbers coincide*:

(i) $(\omega, f)_P$.

(ii) $N(R_\upsilon, \omega)$.

(iii) $\sum_{i=0}^{\infty} [\upsilon_{P_i}(\omega^{(i)}) + \varepsilon^{(i)}]\,\upsilon^*(m_i)$. (*Identifying* $\operatorname{ord}_t(0) = (1, 0)$.)

Let us suppose that f is not a separatrix of ω, hence there exists $\psi \in \hat{R}$, $\psi \neq f$ such that $\omega \wedge df = \psi\eta$ (η basis of $\bar{\Omega}^2_{\hat{R}}$). Then, we have the following proposition.

Proposition 3.7. *Let c be the degree of the conductor of the integral closure of f. Then $N(R_\upsilon, \omega) = (\psi, f) - c$, where (ψ, f) is the intersection multiplicity between the curves f and ψ in P.*

Proof. The value (ψ, f) does not depend on the representatives of the curve and foliation. Let us see the evolution of (ψ, f) under quadratic transformations. Let $\{x, y\}$ be an rsp of R and $\{x', y'\}$ an rsp of $\mathcal{O}_{X^{(1)}, P_1}$, we can suppose $x = x'$, $y = x'y'$; then $\omega \wedge df = \psi(dx \wedge dy)$ and

$$\pi^*(\omega \wedge df) = x'^{\varphi}\omega^{(1)} \wedge df^{(1)} + f^{(1)}(\ldots), \quad \text{for } \varphi = \upsilon_{P_0}(\omega^{(0)}) + \varepsilon^{(0)} + e(f).$$

The term $(\ldots)$ does not affect to the reasoning, and $\omega^{(1)}$, $f^{(1)}$ are the strict transforms of ω and f by π_1, respectively. Thus,

$$\pi_1^*(\psi\,\mathrm{d}x \wedge \mathrm{d}y) = x'\pi_1^*(\psi)\,\mathrm{d}x' \wedge \mathrm{d}y'.$$

If $\omega^{(1)} \wedge \mathrm{d}f^{(1)} = \psi^{(1)}\mathrm{d}x' \wedge \mathrm{d}y'$, one has, $x'^q\psi^{(1)} + f^{(1)}(\ldots) = x'\pi_1^*(\psi)$.

We consider parametrizations $x = x(t)$; $y = y(t)$ of f and $x' = x'(t) = x(t)$; $y' = y'(t) = x(t)/y(t)$ of $f^{(1)}$, replacing and computing ord_t we obtain $e(f)(v_{P_0}(\omega^{(0)}) + \varepsilon^{(0)}) + e(f)) + (\psi^{(1)}, f^{(1)})_{P_1} = e(f) + \mathrm{ord}_t[\psi(x'(t), x'(t)y'(t)] = e(f) + (\psi, f)_{P_0}$. Summarizing

$$(\psi, f)_{P_0} = (\psi^{(1)}, f^{(1)})_{P_1} + e(f)(v_{P_0}(\omega^{(0)}) + \varepsilon^{(0)}) + e(f)(e(f) - 1).$$

If we keep on the process in the remaining infinitely near points of v, (only a finite number of terms is different of zero) we have,

$$(\psi, f)_{P_0} = \sum_{i=0}^{\infty} [v_{P_i}(\omega^{(i)}) + \varepsilon^{(i)}]e(f^{(i)}) + \sum_{i=0}^{\infty} e(f^{(i)})(e(f^{(i)}) - 1).$$

The last sum is c [4, 4.4.5] and the first one is $N(R_v, \omega)$. □

4. Desingularization and valuations

Proposition 4.1. *Let us suppose that the values $v_{P_n}(\omega^{(n)}) + \varepsilon^{(n)}$ are known for any infinitely near point of order n, P_n, of* Spec R *over P_0. Then, we can determine the singularities that appear in the desingularization of ω and their type (i.e., if they are simple or not and if they are dicritical or not).*

Proof. The condition $v_{P_n}(\omega^{(n)}) + \varepsilon^{(n)} = 0$ determines whether P_n is regular. We are going to characterize if P_n is a simple or a dicritical point. For simplicity we shall make it for P_0 (a similar argument can be used for P_n). We may choose P_1 such that $v_{P_1}(\omega^{(1)}) + \varepsilon^{(1)} = 0$, hence P_1 is a regular point of L_1. If we consider P_2 the point of the exceptional divisor L_2 in the direction of L_1, we will have that P_0 is a dicritical point (non-dicritical point, respectively) if P_2 is a regular point (singular point, respectively). In the non-dicritical case P_2 will be simple. Summarizing, L_1 is a dicritical divisor (non-dicritical) if $v_{P_2}(\omega^{(2)}) + \varepsilon^{(2)} = 0$ ($=1$, respectively).

Let L_1 be a non-dicritical divisor. The simple points of L_1 are the singular points of the blown-up foliation $\mathscr{F}^{(1)}$ of ω which possess exactly two transversal separatrices, one of them the exceptional divisor L_1. These points satisfy $v_{P_1}(\omega^{(1)}) = 1$, $\varepsilon^{(1)} = 0$ (equivalently $v_{P_1}(\omega^{(1)}) + \varepsilon^{(1)} = 1$). If P_1 is a simple point, the divisor L_2 contains two simple points, one of them $L_1 \cap L_2$. If P_2 is the other simple point of L_2 and one continues the same process, one has a sequence of infinitely near simple points P_0, $P_1, \ldots, P_n, \ldots$. If P_2 is not a simple point, the condition $v_{P_2}(\omega^{(2)}) = 1$ and the fact that the quotient of the eigenvalues of $dZ_2(P_2)$ (where Z_2 denotes the vector field associated to $\omega^{(2)}$) belongs to $\mathbb{Q}^+$ prove that some of the previous points has to be a dicritical point. (See [3].) Summarizing, P_1 is a simple point if, and only if, $v_{P_1}(\omega^{(1)}) + \varepsilon^{(1)} = 1$, L_2 is not a dicritical divisor and on it there exists a unique point P_2 different from $L_1 \cap L_2$ with $v_{P_2}(\omega^{(2)}) + \varepsilon^{(2)} = 1$ and so on. Alternatively, the non-simple points P_1 with $v_{P_1}(\omega^{(1)}) + \varepsilon^{(1)} = 1$ are those for which there exists a sequence of infinitely near points as above with some P_n dicritic. A similar argument holds for $L_1 \cap L_2$ with algebraic multiplicity 1 (in this case, the points P_n would be the corners with L_1).

If there exists a simple non-corner point, it is obvious that there exists a valuation υ of type B or C such that $(R_\upsilon, \omega) = (1, 0)$. □

Definition 4.2. Let υ and υ' be two divisorial valuations. We say that υ' is *consecutive* to υ if the sequence of infinitely near points associated to υ', $P_0, P_1, \ldots, P_{n+1}$ has the same points as the one associated to υ, $P_0, P_1, \ldots, P_n$, plus another point, P_{n+1}, of L_{n+1}. A finite sequence of consecutive divisorial valuations $\upsilon_0, \upsilon_1, \ldots, \upsilon_n$ (i.e. υ_{i+1} consecutive to υ_i for $0 \le i \le n-1$) is called a *path* of divisorial valuations.

If υ_0 corresponds to the blowing-up in the closed point of Spec R we call the above path a *branch* of divisorial valuations. It is obvious that a path (branch, respectively) is an equivalent data to a finite sequence of infinitely near points that begins in a certain infinitely near point (that begins in P_0, respectively).

Proposition 4.3. *Let $P_0, P_1, \ldots, P_n$ be a sequence of infinitely near points and $\upsilon_0, \upsilon_1, \ldots, \upsilon_n$ the branch of valuations associated to them. The data $\{v_{P_d}(\omega^{(d)}) + \varepsilon^{(d)}\}_{0 \le d \le n}$ and $\{N(R_{\upsilon_d}, \omega)\}_{0 \le d \le n}$ are equivalent.*

Proof. We can write the HNE of each of the previous valuations because we know the sequence of infinitely near points associated to them. So, if $\{x, y\}$ is an rsp of m, the HNE for υ_0 will be $y = xU$; the one of υ_1, $y = a_{01}x + x^2U$, where a_{01} is determinate by P_1. Iterating this process we obtain the HNE, D, for υ_n (see Section 2.2).

On the other hand, the Noether formula:

$$N(R_{\upsilon_d}, \omega) = \sum_{j=0}^{q} (v_{P_j}(\omega^{(j)}) + \varepsilon^{(j)} \upsilon_d(m_j)), \quad d = 0, 1, \ldots, n, \ \upsilon_d(m_q) = 1,$$

shows the equivalence of the above data because the values $\upsilon_d(m_j)$ only depend on the branch and, as we shall prove in (a) and (b), they can be deduced from the HNE of each valuation υ_d.

4.3(a). Let us suppose that the HNE of the valuation υ_d in $\{x, y\}$, an rsp of m, is D (see Section 2.2) such that $a_{s_g h_{s_g}} \ne 0$, then υ_d has associated $q + 1 = \sum_{i=0}^{s_g} h_i + 1$ maximal ideals m_i of $\mathcal{O}_{X^{(i)}, P_i}$. Let us put $\alpha_0 = 1$ and for $i = 1, 2, \ldots, s_g$:

$$\alpha_i = \begin{cases} h_{s_g - (i-1)} & \text{if } i \ne s_g - s_j + 1 \quad \text{for all } j = 0, \ldots, g, \\ k_j & \text{if } i = s_g - s_j + 1 \quad \text{for some } j = 0, \ldots, g, \end{cases}$$

and

$$\beta_i = \begin{cases} 1 & \text{if } i \neq s_g - s_j + 1 \quad \text{for all } j = 0, \ldots, g, \\ 0 & \text{if } i = s_g - s_j + 1 \quad \text{for some } j = 0, \ldots, g. \end{cases}$$

The recurrent formulae $\mathscr{a}_{-1} = \mathscr{a}_{-1} = 0$; $\mathscr{a}_0 = 1$; $\mathscr{a}_i = \alpha_i \mathscr{a}_{i-1} + \beta_i \mathscr{a}_{i-2}, i = 1, 2, \ldots, s_g$ show the values $v_d(m_j)$ as follows:

- The $h_{s_g} + 1$ maximal ideals $m_q, m_{q-1}, \ldots, m_{q-h_{s_g}} = m_{h_{s_g-1} + h_{s_g-2} + \cdots + h_0}$ have value $\mathscr{a}_0$.
- The h_{s_g-1} maximal ideals $m_{q-h_{s_g}-1}, \ldots, m_{q-(h_{s_g}+h_{s_g-1})}$ have value $\mathscr{a}_1$.

$\vdots$

- The h_0 maximal ideals $m_{q-(h_{s_g}+h_{s_g-1}+\cdots+h_1)-1}, \ldots, m_0$ have value $\mathscr{a}_{s_g}$.

If in the HNE, D, the last row is $z_{s_g-1} = z_{s_g}^{h_{s_g}+1} U$, one has the same formulae for $h_{s_g} + 1 = k_g$. Then 4.3(a) is proved.

In (b) we give a way to compute the value $\mathscr{a}_i$ directly.

4.3(b). Let $(\alpha_n)_{n \geq 2}$ and $(\beta_n)_{n \geq 2}$ be two sequences of real numbers and let us consider the recurrent equation $\mathscr{a}_n = \alpha_n \mathscr{a}_{n-1} + \beta_n \mathscr{a}_{n-2}$ $n \geq 2$, $\mathscr{a}_1, \mathscr{a}_0$ fixed real numbers. Then, we have

$$\mathscr{a}_n = \left(\sum_{A \in I_n^*} (\alpha, \beta)_A \right) \mathscr{a}_1 + \beta_2 \left(\sum_{A \in I_n'^*} (\alpha, \beta)_A \right) \mathscr{a}_0,$$

with the following notations:

For $I \subseteq \mathbb{N} - \{0, 1\}$, finite, we denote $I' = I - \{a_0\}$ with $a_0 = \min\{a \mid a \in I\}$ and we put $I^* = \{(A_1, A_2) \mid A_1 \cup A_2 = A \subseteq I$ with $A_1 \subseteq I$; $A_2 \subseteq I'$; $A_1 \cap A_2 = \emptyset$ and such that $\{i - 1 \mid i \in A_2\} = I - A\}$. $(\alpha, \beta)_A = \prod_{i \in A_1, j \in A_2} \alpha_i \beta_j$ and $I_n = \{2, 3, \ldots, n\}$. Let us prove now (b).

We shall use induction on $n \in \mathbb{N}$. If $n = 2$, the proof is trivial. Let suppose that it is true for $p \leq n - 1$. From the recurrent equation $\mathscr{a}_n = \alpha_n \mathscr{a}_{n-1} + \beta_n \mathscr{a}_{n-2}$ and from the inductive expression of $\mathscr{a}_{n-1}$ and $\mathscr{a}_{n-2}$ we have

$$\begin{aligned} \mathscr{a}_n &= \alpha_n \left(\left(\sum_{A \in I_{n-1}^*} (\alpha, \beta)_A \right) \mathscr{a}_1 + \beta_2 \left(\sum_{A \in I_{n-1}'^*} (\alpha, \beta)_A \right) \mathscr{a}_0 \right) \\ &\quad + \beta_n \left(\left(\sum_{A \in I_{n-2}^*} (\alpha, \beta)_A \right) \mathscr{a}_1 + \beta_2 \left(\sum_{A \in I_{n-2}'^*} (\alpha, \beta)_A \right) \mathscr{a}_0 \right) \\ &= \left(\alpha_n \left(\sum_{A \in I_{n-1}^*} (\alpha, \beta)_A \right) + \beta_n \left(\sum_{A \in I_{n-2}^*} (\alpha, \beta)_A \right) \right) \mathscr{a}_1 \\ &\quad + \left(\alpha_n \beta_2 \left(\sum_{A \in I_{n-1}'^*} (\alpha, \beta)_A \right) + \beta_n \beta_2 \left(\sum_{A \in I_{n-2}'^*} (\alpha, \beta)_A \right) \right) \mathscr{a}_0. \end{aligned}$$

From the definition of $(\alpha, \beta)_A$, I^* and I'^* we are led to the result.
This completes the proof of Proposition 4.3. $\square$

Let us remark that in the expression for α_n in the above proof the number of terms that appear in the coefficient of α_1 (respectively, α_0) is the n-term (respectively, $(n-1)$-term) of the Fibonacci sequence, $\{c_s\}$ ($c_s = c_{s-1} + c_{s-2}$, $c_0 = 0$, $c_1 = 1$).

Remark. A somewhat different treatment of the calculations of 4.3(a), (b) is contained in [10] (cf. identities (1.2)–(1.6) and the following explicit formula for $P_n(a_1, \dots, a_n)$, Lemmas 8.1–8.2, Theorem 8.3 and Corollary 8.4).

To accomplish our objective, we give a method to find the singular points that appear after a quadratic transformation. Since there are a finite number of them, we shall know their type by means of Propositions 4.1 and 4.3.

Proposition 4.4. *Let P_n be an infinitely near point of order n of* Spec R *over P_0 such that it is a singular point of the strict transform of ω, $\omega^{(n)}$. Let $\{x, y\}$ be an* rsp *of* m. *Then, the coordinates of the singular free points P_{n+1} of L_{n+1} (in a chart) are given by the common roots, that correspond to free points of L_{n+1}, of two polynomials $P(a)$ and $Q(a)$ with coefficients in K ($Q(a) \equiv 0$ if P_n is not a dicritical point).*

Proof. Suppose that $\omega^{(n)}$ is obtained after blowing-up of $\omega, \omega^{(1)}, \dots, \omega^{(n-1)}$ in the infinitely near points $P_0, P_1, \dots, P_{n-1}$ respectively. Let υ be the divisorial valuation associated to the ssqt $P_0, P_1, \dots, P_n$. One has: $N(R_\upsilon, \omega) = \sum_{i=0}^{n} (v_{P_i}(\omega^{(i)}) + \varepsilon^{(i)})\upsilon(m_i)$. Consider the valuation υ' consecutive to υ associated to $P_0, P_1, \dots, P_{n+1}$. Apart from the maximal ideals of the Noether formula for $N(R_\upsilon, \omega)$, there appears one more maximal ideal m_{n+1}, of $\mathcal{O}_{X^{(n+1)}, P_{n+1}}$, in the case of $N(R_{\upsilon'}, \omega)$. According to Proposition 4.3 one has $\upsilon(m_i) = \upsilon'(m_i)$, for $i \le n$; and $\upsilon'(m_{n+1}) = \upsilon'(m_n) = 1$ since P_{n+1} is a free point of L_{n+1}. Also, we have

$$N(R_{\upsilon'}, \omega) - N(R_\upsilon, \omega) = v_{P_{n+1}}(\omega^{(n+1)}) + \varepsilon^{(n+1)}.$$

In short, a free point P_{n+1} is singular if and only if $N(R_{\upsilon'}, \omega) > N(R_\upsilon, \omega)$. To obtain all the singular points in L_{n+1} we have to study whether the satellite points of L_{n+1} are singular (see Propositions 4.1 and 4.3).

Firstly, we will find the singular free points of the transformed foliation $\mathscr{F}^{(1)}$ of $\mathscr{F}$ under blowing-up in P. Assume that $\omega = a(x, y)\,\mathrm{d}x + b(x, y)\,\mathrm{d}y$ with $a(x, y) = \sum_{i \ge r} a_i(x, y)$, $b(x, y) = \sum_{i \ge r} b_i(x, y)$, where $a_i(x, y)$ and $b_i(x, y)$ are the homogeneous components of $a(x, y)$ and $b(x, y)$ of degree i respectively. Let $\omega^{(1)}$ the germ of $\mathscr{F}^{(1)}$ in a free point P_1 of L_1 whose coordinates in a chart are given by $a \in K$. The HNE of the associated valuation to P_0, P_1 will be $y = ax + x^2 U$. Using Theorem 3.4(iv) one has $(D, \omega) = v_{P_0}(\omega^{(0)}) + \varepsilon^{(0)} + (D^{(1)}, \omega^{(1)})$ ($v_{P_0}(\omega^{(0)}) = r$), and P_1 being a free point, it is singular if and only if $(D, \omega) > v_{P_0}(\omega^{(0)}) + \varepsilon^{(0)}$. The parametrization $x = t$, $y = at + \alpha t^2 + \beta t^2$, gives us

$$
\begin{aligned}
(D,\omega) = \operatorname{ord}_t \omega(D^a) &= \operatorname{ord}_t([a_r(t, at + \alpha t^2 + \beta t^3) + a_{r+1}(t, at + \alpha t^2 + \beta t^3) + \cdots] \\
&\qquad + [b_r(t, at + \alpha t^2 + \beta t^3) + b_{r+1}(t, at + \alpha t^2 + \beta t^3) \\
&\qquad + \cdots](a + 2\alpha t + 3\beta t^2)) \\
&= \operatorname{ord}_t t^r([a_r(1, a + \alpha t + \beta t^2) + t a_{r+1}(1, a + \alpha t + \beta t^2) + \cdots] \\
&\qquad + a[b_r(1, a + \alpha t + \beta t^2) + t b_{r+1}(1, a + \alpha t + \beta t^2) + \cdots] \\
&\qquad + 2\alpha t[b_r(1, a + \alpha t + \beta t^2) + t b_{r+1}(1, a + \alpha t + \beta t^2) + \cdots] \\
&\qquad + 3\beta t^2[b_r(1, a + \alpha t + \beta t^2) + t b_{r+1}(1, a + \alpha t + \beta t^2) + \cdots]).
\end{aligned}
$$

If P_0 is not a dicritical point, we look for values $a \in K$ such that the above ord_t is bigger than r. Namely, those for which the independent term of $a_r(1, a + \alpha t + \beta t^2) + ab_r(1, a + \alpha t + \beta t^2)$ is zero. Therefore, the singularities are given by the roots of a polynomial with coefficients in K, $P(a)$, of degree $p \le r + 1$.

If P_0 is a dicritical point, we know that the formal expression $a_r(1, Y) + Yb_r(1, Y)$ is identically zero and

$$
\begin{aligned}
(D,\omega) = \operatorname{ord}_t t^{r+1}&([a_{r+1}(1, a + \alpha t + \beta t^2) + t a_{r+2}(1, a + \alpha t + \beta t^2) + \cdots] \\
&+ a[b_{r+1}(1, a + \alpha t + \beta t^2) + t b_{r+2}(1, a + \alpha t + \beta t^2) + \cdots] \\
&+ 2\alpha[b_r(1, a + \alpha t + \beta t^2) + t b_{r+1}(1, a + \alpha t + \beta t^2) + \cdots] \\
&+ 3\beta t[b_r(1, a + \alpha t + \beta t^2) + t b_{r+1}(1, a + \alpha t + \beta t^2) + \cdots]).
\end{aligned}
$$

Now, $b_r(1, Y) \not\equiv 0$ implies that the singularities are given by the numbers $a \in K$ such that the independent term of

$$a_{r+1}(1, a + \alpha t + \beta t^2) + ab_{r+1}(1, a + \alpha t + \beta t^2) + 2\alpha b_r(1, a + \alpha t + \beta t^2)$$

vanishes, and so, by the roots in a of an expression $P(a) + \alpha Q(a)$, where $P(a)$ and $Q(a)$ are polynomials with coefficients in K of degrees $p \le r + 1$ and $q \le r$, respectively. Since α is an indeterminate, the values a are the common roots of $P(a)$ and $Q(a)$.

Now consider $\omega^{(n)}$ and the parametrization of υ, $x = x(t, U)$; $y = y(t, U)$ that its HNE D gives to us. We write $p' : R \to K[[t, U]]$ the morphism $p'(x) = x(t, U/t)$, $p'(y) = y(t, U/t)$ that induces $p'_* : \Omega_{R/K} \to \Omega_{K[[t,U]]/K}$ and

$$p'_*(\omega) = t^s \omega^{(n)}, \tag{4.4}$$

with $s = N(R_\upsilon, \omega) - (\upsilon_{P_n}(\omega^{(n)}) + \varepsilon^{(n)})$ and, $\omega^{(n)} = a^{(n)}(t, U)\,\mathrm{d}t + b^{(n)}(t, U)\,\mathrm{d}U$.

If in the HNE of υ we put $U = at + U't^2$, we obtain the parametric expression (derived from its HNE, D^a) of consecutive valuations to υ. We consider only the values of a that correspond to free points of L_{n+1}. Since $\{t, U\}$ is a rsp of the maximal ideal of $\mathcal{O}_{X^{(n)}, P_n}$, this change is the HNE, D'^a, of the valuation of the fraction field of R, F, associated to P_n, P_{n+1}. If in the expression (4.4) we put $at + U't^2$ instead of U, we shall have, $\omega(D^a) = t^s = \omega^{(n)}(D'^a)$. We have proved that the values a which increase the ord_t in the expression $\omega^{(n)}(D'^a)$ are the common roots of two polynomials $P(a)$ and

$Q(a)$. Thus the same happens for $\omega(D^a)$ and moreover, by the above equality this polynomial can be computed from $\omega(D^a)$. $\square$

Remark 4.5. As a result of the above proof, the polynomials $P(a)$ and $Q(a)$ can be obtained by means of techniques of intersection multiplicity with valuations. It is enough to compute $\omega(D^a)$ where D^a is the HNE of the valuation associated to the ssqt with centers $P_0, P_1, \ldots, P_{n+1}^a$ (P_{n+1}^a is the free point of L_{n+1} whose coordinates are given by the number $a \in K$, in a chart).

Corollary 4.6. *Let* $P(a) = \sum_{i=0}^{p} P_i(a)$ *and* $Q(a) = \sum_{i=0}^{q} Q_i(a)$ *be the polynomials found in* Theorem 4.4, *written as a sum of forms of degree i such that* $P_0(a) \neq 0$ *or* $Q_0(a) \neq 0$. *Then the divisor* L_{n+1} *has free singular points if, and only if, the resultant,* $R(P, Q)$, *of the polynomials* $P(a)$ *and* $Q(a)$ *is zero.* (*Whenever* P_n *is a free point, the condition* $P_0(a) \neq 0$ *or* $Q_0(a) \neq 0$ *is not necessary.*) $\square$

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